

Super quantum mechanics, spacetime and quantum information

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Abstract

In the paper it is shown the connection between spacetime and quantum information using an information spacetime as superspace. The connection of quantum information with anticommuting variables is given. Also a solitonic bag is presented where the information is confined.

1 Some ideas from quantum information and super quantum mechanics

A quantum bit (qubit) is a quantum system with a two-dimensional Hilbert space.

More precisely a qubit is the amount of the information which is contained in a pure quantum state from the two-dimensional Hilbert space \mathcal{H}_2 .

A general superposition state of the qubit is

$$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle, \quad (1)$$

where ψ_0 and ψ_1 are complex numbers, $|0\rangle$ and $|1\rangle$ are kets representing two Boolean states. The superposition state has the propensity to be a 0 or a 1 and $|\psi_0|^2 + |\psi_1|^2 = 1$.

The eq.(1) can be written as

$$|\psi\rangle = \psi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (2)$$

where we labelled $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ two basis states zero and one.

The Clifford algebra relations of the 2×2 Dirac matrices is

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad (3)$$

where

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

We choose the representation

$$\gamma^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (5)$$

and

$$\gamma^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6)$$

where σ are Pauli matrices and $\gamma^5 = \gamma^0\gamma^1$.

We shall now assume, that quantum information is connected with an anticommuting variable. So we obtain from the super quantum mechanics.

The glue between quantum mechanics and supersymmetry was first given by the supersoliton Lagrangian in $(1+1)$ space-time dimensions [1] in 1977

$$L = \frac{1}{2}[(\partial_\mu\phi)^2 - V^2(\phi) + \bar{\psi}(i + V'(\phi))\psi] \quad (7)$$

where ϕ was a solitonic Bose field and ψ was a Fermi field and $V(\phi)$ was some nonlinear potential. Supersymmetric quantum mechanics (super quantum mechanics SQM) follows directly via reduction from $(1+1)$ space-time dimensions to $(0+1)$ dimension [2].

SQM was established by E.Witten [3] and P. Salamonson a J. W. Van Holten [4] and corresponding Hamiltonian SSQM has the form:

$$H = \frac{1}{2}p^2 + \frac{1}{2}V^2(x) - \frac{[\psi^*, \psi]}{2}V'(x) \quad (8)$$

where $[p, x] = -i$, $\{\psi^*, \psi\} = 1$ and $V(x)$ is a potencial, where p, x are Bose operators of impuls and space-coordinate and ψ, ψ^* are Fermi variables.

Following the idea that space-time and information are connected as a fiber space we shall show [5] how it will be work for SQM.

We shall use superfield formalism on the superspace (t, θ, θ^*) , where Bose variable is the time and θ, θ^* are Grassmannian anticommuting variable:

$$\{\theta, \theta\} = \{\theta^*, \theta^*\} = 0, \quad [\theta, t] = 0, \quad \{\theta, \theta^*\} = 1. \quad (9)$$

Supersymmetric transformation has the form:

$$\begin{aligned} t' &= t - i(\theta^* \varepsilon - \varepsilon^* \theta), \\ \theta' &= \theta + \varepsilon, \\ \theta^{*'} &= \theta^* + \varepsilon^*, \end{aligned} \quad (10)$$

and the superalgebra generators are:

$$\begin{aligned} Q &= i\partial_\theta - \theta^* \partial_t, \\ Q^* &= -i\partial_{\theta^*} + \theta \partial_t \end{aligned} \quad (11)$$

We can see that

$$\begin{aligned} \{Q, Q^*\} &= 2i\partial_t = 2H, \\ \{Q, H\} &= 0 \end{aligned} \quad (12)$$

Supercovariant derivatives have the form

$$D_\theta = \partial_\theta - i\theta^* \partial_t, \quad D_{\theta^*} = \partial_{\theta^*} - i\theta \partial_t. \quad (13)$$

We shall define a scalar superfield:

$$\Phi(t, \theta, \theta^*) = \Phi^*(t, \theta, \theta^*),$$

and the expansion in θ, θ^* is:

$$\Phi(t, \theta, \theta^*) = x(t) + i\theta\psi(t) - i\psi^*(t)\theta^* + \theta^*\theta D(t). \quad (14)$$

The coefficients in the expansion (14) are in SQM Bose one-dimensional space variable $x(t)$ and Fermi variable $\psi(t)$ connected with the qubit information. The component D can be interpreted as a potencial, as we shall see.

The variation

$$\begin{aligned}\delta\Phi &= i[\varepsilon^*Q^* + Q\varepsilon, \Phi] \\ &= \delta x(t) + i\theta\delta\psi(t) - i\delta\psi^*(t)\theta^* + \theta^*\theta\delta D(t)\end{aligned}\quad (15)$$

a gives us the supersymmetric transformations as follows:

$$\begin{aligned}i\delta x &= \varepsilon^*\psi^* - \psi\varepsilon, \\ \delta\psi &= -i\varepsilon^*D + \varepsilon^*\dot{x}, \\ \delta D &= \frac{\partial}{\partial t}(\varepsilon\psi + \psi^*\varepsilon^*).\end{aligned}$$

We shall assume that a function $f(\Phi)$ can be expanded as:

$$f(\Phi) = \sum_n a_n \Phi^n \quad (16)$$

and the action has the form:

$$S = \int dt d\theta^* d\theta \left(\frac{1}{2}(D_\theta\Phi)^2 - f(\Phi) \right), \quad (17)$$

because $dt d\theta^* d\theta$ is invariant and we assume only the second order derivatives of the superfield components.

We shall use Berezins integral:

$$\int \theta d\theta = \int \theta^* d\theta^* = 1, \quad \int d\theta = \int d\theta^* = 0,$$

and for the construction of Lagrangian we look for coefficient with $\theta\theta^*$.

From the product:

$$(D_\theta\Phi)^*D_\theta\Phi = (-i\psi^* - \theta D + i\theta\dot{x} + \theta^*\theta\dot{\psi}^*) \times (i\psi - \theta^*D - i\theta^*\dot{x} + \theta\theta^*\dot{\psi})$$

we can see that $\theta\theta^*$ coefficient is

$$(\dot{x}^2 + i(\psi^*\dot{\psi} - \dot{\psi}^*\psi) + D^2).$$

For the coefficient with $\theta\theta^*$ in the expansion of $f(\Phi)$ we get:

$$\begin{aligned}&\left\{ \sum n a_n x^{n-1}(-D) + \sum n(n-1)a_n x^{n-2} \left[\frac{1}{2}(\psi\psi^* - \psi^*\psi) \right] \right\} = \\ &= \left\{ -Df'(x) - \frac{1}{2}[\psi^*, \psi]f'' \right\}.\end{aligned}$$

After integration in the action(??) in variables $\theta\theta^*$, we get:

$$S = \int dt \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} i(\psi^* \dot{\psi} - \dot{\psi}^* \psi) + \frac{1}{2} D^2 + Df'(x) + \frac{[\psi^*, \psi]}{2} f'' \right) . \quad (18)$$

We can eliminate the component $D = -f'(x) = V(x)$, as is usual and so we get the SQM Hamiltonian in the form (8).

2 Supersoliton as a bag for quantum information

In the work [5] we show a model of spacetime and information like a fiber space. Following this idea in every moment in super time (t, θ, θ^*) , we can define a supersoliton field $S(x, \theta)$, which is the solution for example of the super sine-Gordon equation:

$$\frac{i}{2} \bar{D} D S(x, \theta) = -\frac{a}{b} \sin bS(x, \theta) ,$$

where we take the constants $a = 2i\sqrt{\alpha_0}$, $b = \frac{\beta}{2}$, as is usual in sine-Gordon model. The supersoliton has the form:

$$S(x, \theta) = \frac{4}{\beta} \text{tg}^{-1} \exp \left[\sqrt{\alpha_0} x + i \frac{\beta}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}(x)} \bar{\theta} \psi_{\text{cl.}}(x) - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right] . \quad (19)$$

We can see

$$\begin{aligned} S(x, \theta) &= \frac{4}{\beta} \text{tg}^{-1} \left[e^{\sqrt{\alpha_0} x} \left(1 + i \frac{\beta}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}(x)} \bar{\theta} \psi_{\text{cl.}}(x) - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right) \right] \\ &= \frac{4}{\beta} \text{tg}^{-1} e^{\sqrt{\alpha_0} x} + \frac{4}{\beta} \frac{e^{\sqrt{\alpha_0} x}}{1 + e^{2\sqrt{\alpha_0} x}} \left(\frac{i\beta \bar{\theta} \psi_{\text{cl.}}(x)}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}(x)} - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right) . \end{aligned} \quad (20)$$

Because: $\frac{e^{\sqrt{\alpha_0} x}}{1 + e^{2\sqrt{\alpha_0} x}} = \frac{\text{tg} \frac{\beta}{4} \varphi_{\text{cl.}}(x)}{1 + \text{tg}^2 \frac{\beta}{4} \varphi_{\text{cl.}}(x)} = \frac{1}{2} \sin \frac{\beta}{2} \varphi_{\text{cl.}}(x)$,

we get:

$$S(x, \theta) = \varphi_{\text{cl.}}(x) + i\bar{\theta} \psi_{\text{cl.}}(x) + \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta F(x) , \quad (21)$$

where F:

$$F = \frac{2i\sqrt{\alpha_0}}{\beta} \sin \frac{\beta}{2} \varphi_{\text{cl.}} .$$

Supersoliton solution(??) can be obtained as follows: we use $e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$, where $y = \left[\sqrt{\alpha_0} x + i \frac{\beta}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}} \bar{\theta} \psi_{\text{cl.}} - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right]$.

Because θ is anticommuting and we know that $\bar{\psi}_{\text{cl.}} \psi_{\text{cl.}} = 0$, we get only $e^{\sqrt{\alpha_0} x}$, and all other are zero (we use $\bar{\theta} \psi = \bar{\psi} \theta$) and so we obtain:

$$e^{\sqrt{\alpha_0} x} \left(1 + i \frac{\beta}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}} \bar{\theta} \psi_{\text{cl.}} - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right) = \text{def. } z ,$$

where def. z means definition of the z .

Now we expand the $f(z) = \text{arctg } z$

$$\begin{aligned} f(z) &= f(\sigma) + f'(\sigma)(z - \sigma) + f''(\sigma)(z - \sigma)^2 + \dots , \\ o &= e^{\sqrt{\alpha_0} x} , \\ (z - \sigma) &= e^{\sqrt{\alpha_0} x} \left(i \frac{\beta}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}} \bar{\theta} \psi_{\text{cl.}} - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right) . \end{aligned}$$

We can see that from $(z - \sigma)^2$ all is equal zero.

Because $f'(\sigma) = \frac{1}{1+\sigma^2}$, we get

$$f(z) = f(e^{\sqrt{\alpha_0} x}) + \frac{e^{\sqrt{\alpha_0} x}}{1 + e^{2\sqrt{\alpha_0} x}} \left(i \frac{\beta}{2 \sin \frac{\beta}{2} \varphi_{\text{cl.}}} \bar{\theta} \psi_{\text{cl.}} - \frac{\sqrt{\alpha_0}}{2} \bar{\theta} \theta \right) .$$

We can see

$$\varphi_{\text{cl.}} = \frac{4}{\beta} \text{arctg } e^{\sqrt{\alpha_0} x} ,$$

and so $e^{\sqrt{\alpha_0} x} = \text{tg } \frac{\beta}{4} \varphi_{\text{cl.}}$.

So we can write:

$$\frac{e^{\sqrt{\alpha_0} x}}{1 + e^{2\sqrt{\alpha_0} x}} = \frac{\text{tg } \frac{\beta}{4} \varphi_{\text{cl.}}(x)}{1 + \text{tg}^2 \frac{\beta}{4} \varphi_{\text{cl.}}(x)} = \frac{1}{2} \sin \frac{\beta}{2} \varphi_{\text{cl.}}(x) .$$

for the solution $\psi_{\text{cl.}}(x)$ we can see

$$\bar{\psi}_{\text{cl.}} \bar{\psi}_{\text{cl.}} = \psi_{\text{cl.}}^+ \gamma^0 \psi_{\text{cl.}} = C^2 (\cosh \sqrt{\alpha_0} x)^{-2} (1, i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = 0$$

and also

$$\bar{\psi}_{\text{cl.}} \gamma^1 \frac{d}{dx} \psi_{\text{cl.}} = \psi_{\text{cl.}}^+ \gamma^0 \psi_{\text{cl.}} = C^2 (\cosh \sqrt{\alpha_0} x)^{-2} (1, i) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = 0 .$$

It means $\psi_{\text{cl.}}(x)$ play no role for energy:

$$E_{\text{cl.}}(\varphi_{\text{cl.}}, \psi_{\text{cl.}}) = E_{\text{cl.}}(\psi_{\text{cl.}}) .$$

It is something like information field $\psi_{\text{cl.}}(x)$ is in in a bag [1].

3 Conclusions

Here we show another aspects of the connection of the spacetime and quantum information. We show the nontrivial connection of quantum information and spacetime via super quantum mechanics and the supersoliton theory, starting from the idea of the connection information and space time like superspace as a fiber space. It is interesting that such anticommuting information variables play no role in energy of the bag, where are confined.

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